ON THE J-INTEGRAL FOR BI-MATERIAL BODIES

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Consider a homogeneous (linear) elastic body $\mathcal{B}$ in a state of plane strain. Then the Eshelby-Rice conservation law [1,2] asserts that given any region $\mathcal{R}$ in $\mathcal{B}$,

$$\int_{\partial\mathcal{R}} (Wn_i - \sigma_{jk}n_k u_{j,i})ds = 0 \tag{1}$$

where W is the strain energy, σ_{jk} is the stress, u_j is the displacement, and n_j is the outward unit normal to the boundary $\partial\mathcal{R}$ of $\mathcal{R}$.

It is our purpose to extend (1) to bi-material bodies. Thus we assume that $\mathcal{B}$ is composed of two homogeneous elastic bodies $\mathcal{B}_1$ and $\mathcal{B}_2$ bonded along the x_1 axis. The boundary conditions at the bond line are balance of forces and continuity of displacements; these require that

$$[\sigma_{j2}] = 0 \tag{2a}$$

$$[u_j] = 0 \tag{2b}$$

where $[f](x_1)$ denotes the jump in a function $f(x_1,x_2)$ across the line $x_2 = 0$

$$[f](x_1) = f(x_1,0^+) - f(x_1,0^-)$$

Note that by (2b) the tangential derivative of u_j must also be continuous across $x_2 = 0$

$$[u_{j,1}] = 0 \tag{3}$$

Now let $\mathcal{R}$ be a region in $\mathcal{B}$ with $\mathcal{R}_\alpha = \mathcal{B}_\alpha \cap \mathcal{R}$ ($\alpha = 1,2$) non-empty (Fig. 1). Then since each $\mathcal{R}_\alpha$ is composed of a single material, (1) holds with $\mathcal{R}$ replaced by $\mathcal{R}_\alpha$. The boundary $\partial\mathcal{R}_\alpha$ of $\mathcal{R}_\alpha$ consists of two parts: a portion $\partial\mathcal{R}_\alpha \cap \partial\mathcal{R}$ that lies on the boundary of $\mathcal{R}$ and the remainder ℓ (= $\mathcal{R}_1 \cap \mathcal{R}_2$) which lies on the line $x_2 = 0$. Thus if we add (1) with $\mathcal{R}$ replaced by $\mathcal{R}_1$ to (1) with $\mathcal{R}$ replaced by $\mathcal{R}_2$, we arrive at

$$\int_{\partial\mathcal{R}} (Wn_i - \sigma_{jk}n_k u_{j,i})ds - \int_{\ell} ([W]\delta_{i2} - \sigma_{k2}[u_{k,i}])ds = 0 \tag{4}$$

where we have used (2a). This is the general conservation law for bi-materials. Most important, however, is the form this equation takes for $i = 1$; indeed, by (3),

$$\int_{\partial R} (Wn_1 - \sigma_{jk}n_k u_{j,1})ds = 0 \tag{5}$$

which is the same as the analogous result for a single material.

Consider now a sharp crack along the bond line as shown in Fig. 2. Then the conservation law (5) asserts that the integral

$$J = \int_{\Gamma} (Wn_1 - \sigma_{jk}n_k u_{j,1})ds \tag{6}$$

is independent of the path Γ surrounding the crack tip. *Thus the standard J-integral of fracture mechanics extends without change to bi-materials provided the bond line is straight.* When the bond line is curved this extension is not valid, since for this case the conservation law (5) will generally involve an integral along ℓ. These conclusions are completely consistent with - and are, in fact, motivated by - the well known result that the J-integral for a single material is valid for an inhomogeneous body provided the inhomogeneity is confined to the x_2 direction ([3], p, 210).

It is not difficult to show, in the usual manner [4], that J for a bi-material is equal to the energy release rate of the crack. Further, using the near-tip stress fields established by Rice and Sih [5] and choosing Γ to be a small circle centered at the crack tip, it can be shown that J is related to the stress intensity factors k_1 and k_2 of [5] by the relation

$$J = (\pi/16)(\lambda_1 + \lambda_2)(k_1^2 + k_2^2) \tag{7}$$

with

$$\lambda_\alpha = \begin{cases} 4(1 - \nu_\alpha)/\mu_\alpha & \text{plane strain} \\ 4/[\mu_\alpha(1 + \nu_\alpha)] & \text{plane stress} \end{cases}$$

μ_α and ν_α being the shear modulus and Poisson's ratio for material α. The right side of (7) is the energy release rate given by Malyshev and Salganik [6]; see also Salganik [7].

We remark that the above results are valid also within the nonlinear theory of elasticity, provided W and σ_{jk} are suitably defined (see, e.g., Knowles and Sternberg [8]). Finally (within the linear theory), the conservation law (1.8) of [8] extends without change to bi-materials under the assumptions underlying (5).

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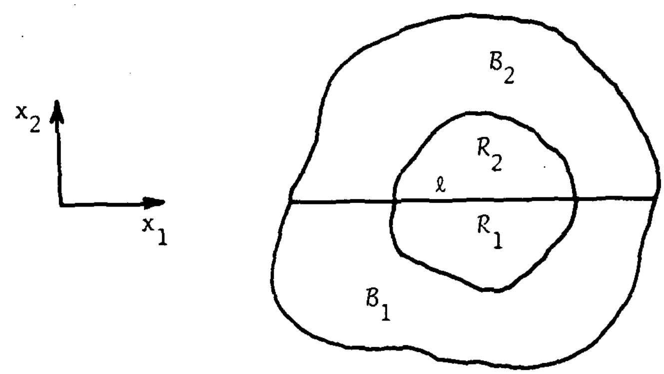

Figure 1

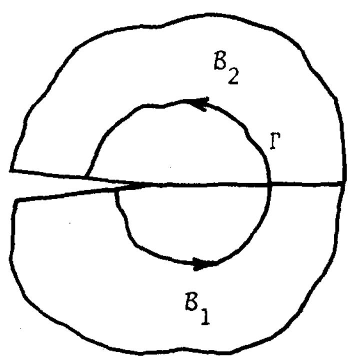

Figure 2